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Lower Bounds for the Partitioning of Graphs

Abstract: Let a k-partition of a graph be a division of the vertices into k disjoint subsets containing $m_1 \geq m_2, \cdots, \geq m_k$ vertices. Let E_c be the number of edges whose two vertices belong to different subsets. Let $\lambda_1 \geq \lambda_2, \cdots, \geq \lambda_k$ be the k largest eigenvalues of a matrix, which is the sum of the adjacency matrix of the graph plus any diagonal matrix U such that the sum of all the elements of the sum matrix is zero. Then

$$E_c \geq \tfrac{1}{2} \sum_{r=1}^{k} - m_r \lambda_r.$$

A theorem is given that shows the effect of the maximum degree of any node being limited, and it is also shown that the right-hand side is a concave function of U. Computational studies are made of the ratio of upper bound to lower bound for the two-partition of a number of random graphs having up to 100 nodes.

Introduction

Partitioning of graphs occurs in computer logic partitioning [1, 2], paging of computer programs [3, 4], and may also find application in the area of classification [5]. Graph partitioning is the problem of dividing the vertices of a graph into a given number of disjoint subsets such that the number of nodes in each subset is less than a given number, while the number of *cut* edges, i.e., edges connecting nodes in different subsets, is a minimum. The problem of computer logic partitioning is actually somewhat different; for a thorough description of that problem, see Ref. 1. The partitioning of graphs is a simplified version of that problem.

In this paper, we assume that the number of vertices in each subset is prescribed. Let $A = A(G)$ be the adjacency matrix of the graph G, which will be defined later, and U any diagonal matrix with the property that trace (U) is the negative of the sum of the valences of the vertices. We derive in Theorem 1 a lower bound for the number of cut edges in terms of the eigenvalues of $A + U$. For the case of division into two subsets, we present, using a different method of derivation, another bound that is stricter for this special case. The bound given in Theorem 1 turns out to be a concave function of U, a fact that suggests exploitation by means of mathematical programming. Computational results are presented, in which the bound is compared with the results of actual, but not necessarily minimal, partitioning. We also compare experimentally the results when $U_{ii} = -d_i$ (the valence of vertex i), and when the $\{U_{ii}\}$ vary.

We believe that, in combinational problems whose complexity suggests the use of heuristic methods, such as the partitioning of graphs, it is worthwhile to have a lower bound on what can be achieved, regardless of the algorithm, provided the calculation of the bound is itself not too onerous and the bounds derived are not too far from the correct value. The results presented here may satisfy these criteria. The calculation of the bound may itself suggest new approaches to the original problem. Also, the fact that different methods are used to derive the bounds of Theorems 1 and 2 suggests that a more comprehensive approach to the problem may be possible. We should also mention that a different use of eigenvalues and eigenvectors on a related problem is discussed in Ref. 6.

This paper does not present details of our experiments on the algorithm for varying U, because a new method which converges to the maximum value of the bound has since been found by Jane Cullum. We are grateful to Jane Cullum and Philip Wolfe, of the Watson Research Center, for useful conversations about the present work.

Derivation of lower bound

Let G be a graph, with edge set E, and vertex set V. For any set S, $|S|$ denotes the number of elements in S. Let $A(G) = (a_{ij})$ be a square matrix of order $|V|$ and be defined by:

$$a_{ij} = \begin{cases} 1 \text{ if vertices } i \text{ and } j \text{ are joined by an edge,} \\ 0 \text{ otherwise.} \end{cases}$$

Thus, $A(G)$ – the adjacency matrix of G – is a square symmetric (0, 1) matrix with 0 diagonal.

Let the eigenvalues of any real symmetric matrix M be denoted by $\lambda_1(M) \geq \lambda_2(M) \geq \cdots$; let U be any diagonal matrix such that $\Sigma_i U_{ii} = -2|E|$; let $m_1 \geq m_2 \geq \cdots \geq m_k$ be given positive integers such that $\Sigma m_i = |V|$; and let $V_1, \cdots, V_k$ be disjoint subsets of V such that $|V_i| = m_i$, $i = 1, \cdots, k$. Finally, let E_c be the set of edges of G, each of which has its two endpoints in different V_i.

Theorem 1. Given the notation above,

$$|E_c| \geq -\frac{1}{2} \sum_1^k m_i \lambda_i (A + U).$$

The right-hand side is a concave function of U.

Proof. It is easy to see that the main theorem of Hoffman and Wielandt [7], when applied to real symmetric matrices M and N of order n, yields

$$\text{Trace } MN^{\mathrm{T}} \leq \sum_1^n \lambda_i(M) \lambda_i(N), \tag{1}$$

in which the Trace of a matrix denotes the sum of all the elements of the diagonal. We note that, if N^{T} is the transpose of N, then Trace $MN^{\mathrm{T}} = \Sigma_{ij} M_{ij} N_{ij}$. Let $M = A + U$ and N be the direct sum of k matrices, each of which consists entirely of 1's, and is defined on the rows and columns corresponding to $V_i (i = 1, \cdots, k)$. Then

$$\lambda_1(N) = m_1, \cdots, \lambda_k(N) = m_k,$$

$$\lambda_{k+1}(N) = \cdots = \lambda_{|V|}(N) = 0. \tag{2}$$

It is clear that

$$\sum_1^{|V|} \lambda_i(M) \lambda_i(N) = \sum m_i \lambda_i (A + U). \tag{3}$$

On the other hand,

$$\text{Trace } MN^{\mathrm{T}} = -2|E| + 2(|E| - |E_c|) = -2|E_c|. \tag{4}$$

Inserting (3) and (4) into (1) proves the first sentence of Theorem 1.

To prove the second sentence, it is sufficient to show that $\Sigma m_i \lambda_i (A + U)$ is a convex function of U. If R, S are real symmetric matrices of order n, and if $l \leq n$, then [8]

$$\sum_1^l \lambda_i (R + S) \leq \sum_1^l \lambda_i(R) + \sum_1^l \lambda_i(S).$$

Hence

$$\begin{aligned} \sum_1^l \lambda_i \left[A + \frac{1}{2}(U_1 + U_2)\right] &= \sum_1^l \lambda_i \left[\frac{1}{2}(A + U_1) + \frac{1}{2}(A + U_2)\right] \\ &\leq \sum_1^l \frac{1}{2}(A + U_1) + \sum_1^l \frac{1}{2} \lambda_i (A + U_2). \end{aligned}$$

$\sum_1^l \lambda_i (A + U)$ is a convex function of U.

Next

$$\begin{aligned} \sum_1^k m_i \lambda_i (A + U) &= (m_1 - m_2) \lambda_1 (A + U) \\ &+ (m_2 - m_3)[\lambda_1(A + U) + \lambda_2(A + U)] \\ &+ \cdots + m_k [\lambda_1 (A + U) \\ &+ \cdots + \lambda_k (A + U)]. \end{aligned} \tag{5}$$

Since $m_i \geq m_{i+1}$, $i = 1, \cdots, k - 1$, and $m_k > 0$, it follows that the right-hand side of (5) is a nonnegative sum of convex functions of U and hence a convex function of U.

The next theorem is concerned with a partition into two equal groups when the maximum degree of any vertex of G is less than d.

Theorem 2. Given 1) a graph with an even number of vertices, 2) that $m_1 = m_2 = |V|/2$, 3) that the degree of any vertex does not exceed some value d, and 4) that $0 \leq \delta_1 \leq \pi/4$, $0 \leq \delta_2 \leq \pi/4$, and 5) that $x \leq 1$ represents a simultaneous solution of the equations

$$x \sin 2\delta_1 = (1 - x) \sin 2\delta_2, \tag{6}$$

$$-[\lambda_1(A + U) + \lambda_2(A + U)]/2 = x\{1 - \sin 2\delta_1 + 2(d - 1)[1 - \cos(\delta_1 + \delta_2)]\}, \tag{7}$$

then

$$|E_c| \geq x|V|/2. \tag{8}$$

Note: Setting $\delta_1 = \delta_2 = 0$ causes this theorem to be a special case of Theorem 1, namely, the case in which $m_1 = m_2 = |V|/2$.

Proof. We first show that if there exists any partition into two equalized groups with $e < |V|/2$ edges cut, then

$$-\frac{1}{2}[\lambda_1(A + U) + \lambda_2(A + U)] \leq Z\{1 - \sin 2\alpha_1 + 2(d - 1)[1 - \cos(\alpha_1 + \alpha_2)]\}, \tag{9}$$

where $Z = 2e/|V|$ and α_1 and α_2 are any numbers satisfying

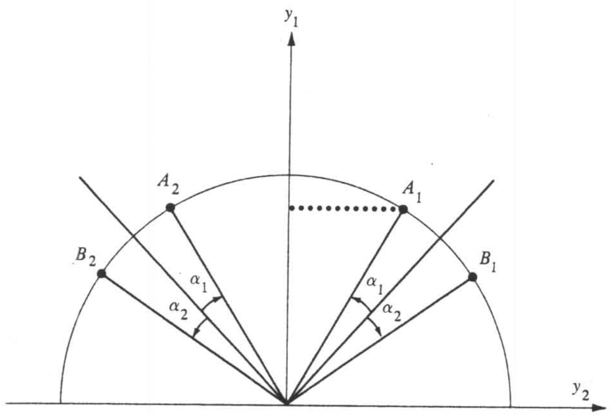

Figure 1 Locations of the groups A_1, A_2, B_1, B_2 in a coordinate system defined by y_1, y_2. These four groups are defined in the proof of Theorem 2.

$$Z \sin 2\alpha_1 = (1 - Z) \sin 2\alpha_2,\ 0 \le \alpha_1, \alpha_2 \le \pi/4. \tag{10}$$

Later we show that the result given above is sufficient to prove the theorem.

It has been shown [9] that, if y_1 and y_2 are any two orthonormal vectors, then

$$\lambda_1(A + U) + \lambda_2(A + U) \ge y_1^{\mathrm{T}}(A + U)y_1 + y_2^{\mathrm{T}}(A + U)y_2, \tag{11}$$

since A and U are symmetric matrices. We can furthermore see that

$$y_k^{\mathrm{T}}(A + U)y_k = \sum_i \sum_j A_{ij} y_{ki} y_{kj} + \sum_i U_{ii} y_{ki}^2 = -\frac{1}{2} \sum_i \sum_j A_{ij}(y_{ki} - y_{kj})^2 + \sum_i (U_{ii} + \sum_j A_{ij}) y_{ki}^2, \tag{12}$$

where y_{ki} are the components of y_k. Let us define

$$U_{ii}' = U_{ii} + \sum_j A_{ij}. \tag{13}$$

Since $\sum_i \sum_j A_{ij} + \sum_i U_{ii} = 0$, we have

$$\sum_i U_{ii}' = 0. \tag{14}$$

We now divide further each of the two groups 1, 2 into which the corresponding V that was partitioned; subgroup $A_k (k = 1,2)$ is a set of exactly e vertices, which includes *all* the vertices that have connections to vertices not belonging to group k. As long as $e < |V|/2$, such a set can always be generated. The other subgroup B_k has, of course, only connections to group A_k. There are $|V|/2 - e$ vertices in B_k, and N_k connections between A_k and B_k. We now set values for $y_{mi} (m = 1,2)$, as indicated in Fig. 1.

$$
\begin{aligned}
i \in B_1 \quad & y_{1i} = \sqrt{2/|V|} \cos (\pi/4 + \alpha_2) \\
& y_{2i} = \sqrt{2/|V|} \sin (\pi/4 + \alpha_2) \\
i \in A_1 \quad & y_{1i} = \sqrt{2/|V|} \cos (\pi/4 - \alpha_1) \\
& y_{2i} = \sqrt{2/|V|} \sin (\pi/4 - \alpha_1) \\
i \in A_2 \quad & y_{1i} = \sqrt{2/|V|} \cos (\pi/4 - \alpha_1) \\
& y_{2i} = -\sqrt{2/|V|} \sin (\pi/4 - \alpha_1) \\
i \in B_2 \quad & y_{1i} = \sqrt{2/|V|} \cos (\pi/4 + \alpha_2) \\
& y_{2i} = -\sqrt{2/|V|} \sin (\pi/4 + \alpha_2).
\end{aligned} \tag{15}
$$

Since $|A_1| = |A_2|$ and $|B_1| = |B_2|$, then y_1 and y_2 are clearly orthogonal. We now show that condition (10) proves $\|y_k\| = 1,\ k = 1, 2$. It can be seen that

$$y_1^{\mathrm{T}} y_1 = (2/|V|)[(|V| - 2e) \sin^2 (\pi/4 + \alpha_2) + 2e \sin^2 (\pi/4 - \alpha_1)],$$

$$y_2^{\mathrm{T}} y_2 = (2/|V|)[(|V| - 2e) \cos^2 (\pi/4 + \alpha_2) + 2e \cos^2 (\pi/4 - \alpha_1)],$$

so that $y_1^{\mathrm{T}} y_1 + y_2^{\mathrm{T}} y_2 = 2$. However, we also require for normality that $y_1^{\mathrm{T}} y_1 - y_2^{\mathrm{T}} y_2 = 0$ so that

$$0 = (2/|V|)[(|V| - 2e) \cos (\pi/2 + 2\alpha_2) + 2e \cos (\pi/2 - 2\alpha_1)]$$

or, dividing by two and using $Z = 2e/|V|$, we have

$$0 = (1 - Z) \cos (\pi/2 + 2\alpha_2) + Z \cos (\pi/2 - 2\alpha_1).$$

Since $\cos (\pi/2 + x) = -\sin x$, we find the above to be equivalent to $0 = -(1 - Z) \sin 2\alpha_2 + Z \sin 2\alpha_1$, which is condition (10).

Inserting Eq. (12) into (11) we have

$$-\lambda_1(A + U) - \lambda_2(A + U) \le \frac{1}{2} \sum_i \sum_j A_{ij}[(y_{1i} - y_{1j})^2 + (y_{2i} - y_{2j})^2] - \sum_i U_{ii}'(y_{1i}^2 + y_{2i}^2).$$

However, from Eq. (15) it follows that $y_{1i}^2 + y_{2i}^2 = 2/|V|$ and, when Eq. (14) is used, the term in U_{ii}' falls out. The other part becomes, on substitution of Eq. (15),

$$-\lambda_1 - \lambda_2 \le \sum_{i \in A_1} \sum_{j \in A_2} A_{ij}(8/|V|) \sin^2 (\pi/4 - \alpha_1) + \Big(\sum_{i \in A_1} \sum_{j \in B_1} + \sum_{i \in A_2} \sum_{j \in B_2}\Big)(2A_{ij}/|V|)[\sin (\pi/4 - \alpha_1) - \sin (\pi/4 + \alpha_2)]^2 + [\cos (\pi/4 - \alpha_1) - \cos (\pi/4 + \alpha_2)]^2,$$

which simplifies to

$$-\lambda_1 - \lambda_2 \le (8e/|V|)[\sin^2(\pi/4-\alpha_1)]$$
$$+ (2/|V|)(N_1+N_2)[\sin^2(\pi/4-\alpha_1)$$
$$+ \sin^2(\pi/4+\alpha_2)$$
$$- 2\sin(\pi/4-\alpha_1)\sin(\pi/4+\alpha_2)$$
$$+ \cos^2(\pi/4-\alpha_1)$$
$$- 2\cos(\pi/4-\alpha_1)\cos(\pi/4+\alpha_2)$$
$$+ \cos^2(\pi/4+\alpha_2)].$$

Upon using standard geometric identities, we have

$$-\lambda_1 - \lambda_2 \le (4e/|V|)(1-\sin 2\alpha_1) + [4(N_1+N_2)/|V|] \times [1-\cos(\alpha_1+\alpha_2)].$$

Because each node has a maximum degree of d, we have

$$e + N_1 \le ed$$
$$e + N_2 \le ed$$

so that $N_k \le e(d-l)$ and using $Z = 2e/|V|$, we have after also dividing both sides by two

$$\frac{1}{2}(-\lambda_1 - \lambda_2) \le Z(1-\sin 2\alpha_1) + 2Z(d-1) \times [1-\cos(\alpha_1+\alpha_2)],$$

which is the inequality (9).

The second part of the proof consists in showing that any possible value of x that solves Eqs. (6) and (7) must be less than Z. The value of x is then used in the inequality (8).

Let us assume that we have found x, δ_1, δ_2 satisfying Eqs. (6) and (7) and that x exceeds the minimum possible value of Z, which is a characteristic of the graph. We fix $\alpha_1 = \delta_1$, and with $x > Z$, it turns out that α_2 exists if δ_2 exists, and furthermore, that $\alpha_2 < \delta_2$, which can be verified by inspecting conditions (6) and (10). This leads to

$$\alpha_1 + \alpha_2 < \delta_1 + \delta_2 \le \pi/2$$

and

$$-\cos(\alpha_1+\alpha_2) < -\cos(\delta_1+\delta_2),$$

so that, with $d \ge 1$,

$$1 - \sin 2\delta_1 + 2(d-1)\ [1-\cos(\delta_1+\delta_2)$$
$$> 1 - \sin 2\alpha_1 + 2(d-1)[1-\cos(\alpha_1+\alpha_2)].$$

Using Eq. (9) we find

$$x\{1-\sin 2\delta_1 + 2(d-1)[1-\cos(\delta_1+\delta_2)]\}$$
$$> Z\{1-\sin 2\alpha_1 + 2(d-1)[1-\cos(\alpha_1+\alpha_2)]\}$$
$$\ge -\frac{1}{2}(\lambda_1+\lambda_2).$$

The transitivity of the $>$ relationship leads us then to conclude, contrary to hypothesis, that Eq. (7) is not satisfied, so that $x \le Z$.

Q.E.D.

Theorem 2 is interesting in the case for partitions into two groups in which E_c is vanishingly small as compared to $|V|$. In this limit, $\delta_2 \to 0$ and we may readily compute the minimum of $[1-\sin 2\delta_1 + 2(d-1)(1-\cos\delta_1)]$. This allows us to compute the ratios R of the bound given by Theorem 2 to that of Theorem 1 as a function of d:

$d=$	$R=$
3	1.68
4	1.42
5	1.30
10	1.12
20	1.06
50	1.02

This shows that, for small d, the bound given by Theorem 1 will be off by a significant amount. While a factor of two to four between actual result and theoretical bound may be tolerable, since one may be able to develop heuristic rules for such a ratio, much larger factors would make the present work useless. Accordingly, some results are presented in the next section showing that the ratio R is, at least in certain cases, not excessive.

Let $B = A + D$, where D is a diagonal matrix chosen so that each row sum of B is 0; i.e., d_{ii} is the negative of the valence of vertex i. For this choice, $D = U$, we obtain another improvement of Theorem 1. This theorem yields a better estimate if the $\{m_i\}$ are different.

Theorem 3. Let B be defined as above, and $\alpha_2 \ge \cdots \ge \alpha_k$ be the roots of

$$f(x) = \left(\sum m_i\right)x^{k-1} - 2\sum_{i<j} m_i m_j x^{k-2} + 3\sum_{i<j<p} m_i m_j m_p x^{k-3} \mp \cdots = 0. \tag{16}$$

Then

$$E_c \ge -\frac{1}{2}\sum_{j=1}^{k} \alpha_j \lambda_j(B). \tag{17}$$

Proof. Let J be the matrix of 1's, and N be as defined in the proof of Theorem 1. By the methods used in Theorem 1, it follows that

$$E_c \ge -\frac{1}{2}\sum_{j=1}^{n} \lambda_j(tJ+N)\lambda_j(B)\ [\text{because Tr}\ (tJ+N)B$$
$$= \text{Tr}\ NB] = -\frac{1}{2}\sum_{j=2}^{n} \lambda_j(tJ+N)\lambda_j(B),$$

since the hypotheses on B show that $\lambda_1(B) = 0$. Inequality (17) is valid for all $t \to \infty$. But it is easy to see that as $t \to \infty$, $\lambda_1(tJ+N) \to \infty$, $\lambda_2(tJ+N) \to \alpha_2, \cdots,$

Table 1 Computed bounds with partitioning of results into two groups.

Graph number	*No. of nodes*	*No. of edges*	B_L $U_{ii}' = 0$	B_L *Best U'*	B_U (*Heuristic partition*)	B_U/B_L
A1	20	54	7	11	13	1.18
A2	20	51	5	11	13	1.18
A3	20	45	4	7	10	1.43
A4	20	46	6	10	15	1.50
A5	19	45	7	9	12	1.33
A6	20	40	3	7	9	1.29
A7	20	48	5	9	13	1.44
A8	20	34	2	5	7	1.40
A9	20	51	8	13	16	1.23
A10	20	51	5	10	14	1.40
A11	20	46	5	8	11	1.38
A16	20	42	4	9	11	1.22
A13	20	52	4	11	15	1.36
A14	20	43	3	8	11	1.38
A15	20	40	4	6	10	1.67
A16	20	44	4	9	12	1.33
A17	20	54	8	12	17	1.42
A18	20	35	2	5	8	1.80
A19	20	45	6	9	14	1.56
A20	20	42	6	8	13	1.63
					Average $B_U/B_L =$	1.41
B1	40	100	12	15	27	1.80
B2	40	92	8	13	23	1.77
B3	40	104	9	17	25	1.47
B4	50	80	6	9	17	1.89
B5	38	78	5	11	16	1.45
B6	40	91	9	13	21	1.62
B7	39	118	18	22	31	1.41
					Average $B_U/B_L =$	1.63
C1	59	162	13	26	41	1.58
C2	58	153	10	25	40	1.60
C3	60	152	11	24	37	1.54
C4	59	142	13	21	32	1.52
C5	59	147	9	20	33	1.65
					Average $B_U/B_L =$	1.58
D1	99	232	14	28	47	1.68
D2	100	264	21	36	54	1.50
D3	100	252	12	34	58	1.71
D4	100	238	13	30	49	1.63
D5	97	272	19	40	62	1.55
					Average $B_U/B_L =$	1.61

$\lambda_k(tJ+N) \to \alpha_k$, $\lambda_{k+1}(tJ+N) = \cdots = \lambda_n(tJ+N) = 0$. The reason is as follows. The matrix $tJ+N$ is positive semidefinite, and if x is any vector such that, for each $V_i (i=1,\cdots,k)$, $\sum_{j\in V_i} x_j = 0$, then $(tJ+N)x = 0$. It follows that the eigenvectors x corresponding to positive eigenvalues of $tJ+N$ have $x_k = x_\ell$ if $k, \ell \in V_i$. Accordingly, the nonzero eigenvalues of $tJ+M$ are the eigenvalues of the $k \times k$ matrix $N(t)$, where

$$[(N(t)]_{r,s} = \begin{cases} (t+1)m_r & \text{if } r=s \\ tm_s & \text{if } r \neq s \end{cases} \quad r, s = 1, \cdots, h.$$

Clearly $\lambda_1[N(t)] \to \infty$. The other eigenvalues of $N(t)$ approach limits that are the roots of the polynomial that is the coefficient of the highest power of t present in the characteristic polynomial of $N(t)$. The characteristic polynomial of $N(t)$ is $\prod_1^k (x-m_i) - tf(x)$.

To prove that (17) is a better bound than that provided by Theorem 1 in the case $D = U$, it is sufficient to show $\alpha_i \geq m_i$ for $i = 2, \cdots, k$. But $N(t)$ is similar to diag $(m_1, \cdots, m_k) + t(\sqrt{m_i m_j})$. Since $t\sqrt{m_i m_j}$ is positive semidefinite for $t \geq 0$, we have completed the proof.

Computational results

Graphs were generated by connecting a preset number of vertices with some probability p, and removing unconnected vertices from the graph. The lower bound B_L on the number of edges cut by a partition into two equal-sized groups was first computed with $U_{ii} = -\Sigma_j A_{ij}$, and then U was varied using the procedure of the two preceding sections to obtain a "best" U with maximum B_L. A heuristic procedure was then used to obtain a partition into groups with B_U edges cut, which is an upper bound on the minimum number of edges cut by such a partition. Results are given in Table 1 for graphs having 20, 40, 60, and 100 nodes; the ratio B_U/B_L is computed for each graph and averaged over all graphs of each of the various sets of equal size. It can be seen that this ratio which is about 1.6 for many of the cases, gives a reasonable range in view of our Theorem 2.

From the results one can also see that variation of U improves B_L significantly—a factor of two improvement is the rule for the larger graphs.

Lastly, two graphs are given in detail in Tables 2 and 3, together with a partition.

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References

1. R. L. Russo, P. H. Oden and P. K. Wolff, Sr., "A Heuristic Procedure for the Partitioning and Mapping of Computer Logic Blocks to Modules," to be published in the *IEEE Trans. Computers*.

Table 2 The connections and the partition of Graph A1 (see Table 1).

Node	*Connections to*
1	2, 3, 4, 7, 8, 17
2	3, 10, 14, 15, 16
3	8, 12, 16
4	7, 9, 11, 17
5	6, 9, 11, 15, 16, 20
6	7
7	9, 15, 16
8	10, 12, 14, 16, 18
9	12, 20
10	12, 14, 16, 19
11	18, 19, 20
12	13, 15
13	14, 16, 18, 19
14	16, 18, 19
15	16, 17, 19
17	18

Partition into two groups, where 13 edges are cut.
Group 1: 1, 4, 5, 6, 7, 9, 11, 17, 18, 20
Group 2: 2, 3, 8, 10, 12, 13, 14, 15, 16

Table 3 The connections and the partition of Graph A2 (see Table 1).

Node	*Connections to*
1	7, 12, 13, 14, 15, 16, 17
2	12, 17, 18, 20
3	5, 11, 13, 14, 18, 19, 20
4	6, 9
5	7, 9, 10, 12, 16, 19
6	16, 18, 20
7	8, 9, 11, 16
8	15, 18
9	11, 15, 19
11	14, 17, 18, 20
12	14
13	18, 20
14	16, 18, 20
16	18
17	18
18	20

Partition into two groups, where 13 edges are cut.
Group 1: 1, 2, 3, 11, 12, 13, 14, 17, 18, 20
Group 2: 4, 5, 6, 7, 8, 9, 10, 15, 16

2. H. R. Charney and D. L. Plato, "Efficient Partitioning of Components," Share/ACM/IEEE Design Automation Workshop, Washington, D. C., July, 1968.
3. L. W. Comeau, "A Study of User Program Optimization in a Paging System," ACM Symposium on Operating System Principles, Gatlinburg, Tennessee, October, 1967.
4. P. J. Denning, "Virtual Memory," *Computing Surveys* **2,** 153 (September 1970).
5. C. T. Zahn, "Graph Theoretical Methods for Detecting and Describing Gestalt Clusters," *IEEE Trans. Computers* **C-20,** 68 (1971).
6. K. M. Hall, "r-Dimension Quadratic Placement Algorithm," *Management Science* **17,** 219 (1971).
7. A. J. Hoffman and H. W. Wielandt, "The Variation of the Spectrum of a Normal Matrix," *Duke Math. J.* **20,** 37 (1953).
8. M. Marcus and H. Minc, *A Survey of Matrix Theory and Matrix Inequalities*, Allyn and Bacon, Inc., Boston, 1964 p. 120, Chapter II. 4.4.14.
9. K. Fan, "On a Theorem of Weyl Concerning Eigenvalues of Linear Transformations, I," *Proc. National Academy Science USA* **35,** 652 (1949).

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